DIFFRACTION AND FOURIER OPTICS

LAB NOTEBOOK GUIDELINES

The lab notebook will play an important role in this course. You will use your notebook for keeping records of many things including

- Answering pre-lab questions from the lab guide.
- Answering in-lab questions.
- Recording data.
- Including plots of data.
- Analysis and results.
- Diagrams and pictures.
- Procedures of experiments that you design.

The lab notebook will be an important part of your grade because learning to keep a good lab notebook is an important part of your professional development. You may find it helpful to write up many of your notes on the computer, for example, within Mathematica or another program. This is fine. However, before your notebook is turned in, the notes, plots, and analysis should be transferred to the lab notebook by printing and taping the pages or keeping them in a three ring binder. There will also be formal lab reports and oral presentations, but these will be restricted to a limited portion of the experimental work you have conducted in the lab.

WEEK 1

WEEK 1 GOALS

- Build a basic setup to measure the Fresnel and Fraunhofer diffraction pattern.
- Understand the limits of validity of the Fresnel and Fraunhofer diffraction models.
- Develop a quantitatively accurate model of the the diffraction and measurement system.
- Computationally predict 1D diffraction patterns.
- Designing an experiment to test the transition between the Fresnel and Fraunhofer regimes.

FROM THE WAVE EQUATION TO A BASIC DIFFRACTION INTEGRAL

Diffraction is phenomena that occurs for any kind of wave, and refers to the propagation of a wave as it goes past an obstacle or through an aperture.

Often in optics it is sufficient to model light using a scalar wave equation. This valid when we ignore polarization. [Say more]

$$\nabla^2 \vec{\boldsymbol{E}} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} \tag{1}$$

Becomes letting ψ be the scalar field for the electric field

$$\nabla^2 \psi = \mu_0 \epsilon_0 \frac{\partial^2 \psi}{\partial t^2} \tag{2}$$

There is an approximate solution to the wave equation given by

$$\psi_{diff}(\vec{R}) = \frac{1}{i\lambda} \int \psi_{trans}(x, y) \frac{\exp ikr_p}{r_p} dx\, dy \tag{3}$$

Where the integral is over the plane of the aperture, and $\psi_{trans}(x, y)$ is the fraction of the field that transmits through the aperture. Note that $\psi_{trans}(x, y) = 0$ for any points x and y that are blocked by the aperature. The distance r_p is the distance between the point (x, y, z) in the aperature plane, and the point (X, Y, Z) in the plane (of your choosing) where we are calculating and observing the diffraction pattern. Figure 1 shows the setup for the integral, and is interpreted in the following way. Each point in the aperture plane acts like a new source of waves propagating out to the diffraction plane. The integral sums up all the contributions to get the electric field at a position (X, Y, Z) in the diffraction plane.

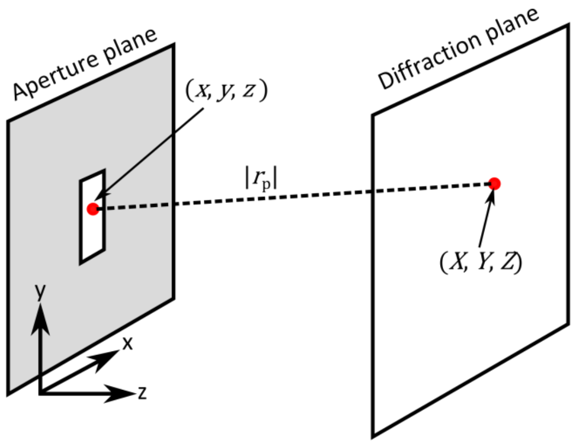


Figure 1: Diagram explaining parameters in the Huygens-Fresnel diffraction integral.

Question 0	**Assemble your setup. (more prelab questions follow)** As soon as you get a chance, assemble a basic diffraction setup using a HeNe laser and an adjustable width slit for the aperture. a. Do you see a diffraction pattern? b. How does the pattern depend on slit width? c. Does it change with distance to the slit? The remainder of this lab will build a quantitative model of diffraction and allow you to do detailed comparison of your model and the data.
Question 1	**Building a beam expander** Throughout this lab it will be important to know the diameter of your beam as it is incident upon the aperture. You may find it necessary to control the size of the beam as well, for example to enlarge it for more uniform illumination. A “beam expander” is a combination of a short and long focal length lens to expand the beam. a. If the lenses have focal lengths f_1 and f_2, how far should the lenses be separated so that incident collimated light leaves the “beam expander” collimated? b. How does the expansion ratio relate to the focal lengths f_1 and f_2? c. Build a 2x beam expander.

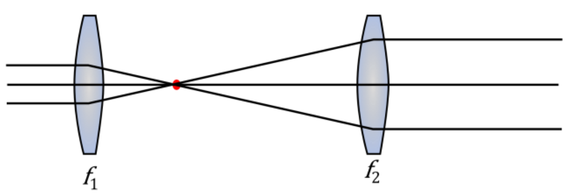


Figure 2: Two lenses form a beam expander.

FRESNEL AND FRAUNHOFER APPROXIMATIONS

The Huygens integral given in equation (3) can undergo additional simplification as follows:

First, the diffraction plane is assumed to be a long ways from the slit compared to the slit width, so

$$\frac{1}{r_p} \approx \frac{1}{z_p} \tag{4}$$

Where $z_p = Z - z$ is the distance between the aperture plane and the diffraction plane.

Next, we approximate $kr_p = k\sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}$

In the Fresnel approximation we take $|Z - z| \gg |X - x|$ and $|Z - z| \gg |Y - y|$ so that we expand

$$\begin{aligned} kr_p &= k\sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2} \\ &= kz_p\sqrt{1 + \frac{(X-x)^2}{z_p^2} + \frac{(Y-y)^2}{z_p^2}} \\ &\approx kz_p\left(1 + \frac{1}{2}\frac{(X-x)^2}{z_p^2} + \frac{1}{2}\frac{(Y-y)^2}{z_p^2}\right) \\ &= kz_p + \frac{k(X^2+Y^2)}{2z_p} + \frac{k(-2xX - 2yY + x^2 + y^2)}{2z_p} \end{aligned} \tag{5}$$

The first two terms in the approximation don't depend on the position (x, y) in the aperture plane, so they are constants in the integral. The third term is the part that contains all the interference effects from the different Huygens waves propagating out from the aperture.

The full Fresnel diffraction integral becomes

$$\begin{aligned} &\psi_{diff}^{\text{Fresnel}}(\vec{R}) \\ &= \frac{1}{i\lambda z_p}\exp(ikz_p)\exp\left(ik\frac{X^2+Y^2}{2z_p}\right)\int \psi_{trans}(x,y)\exp\left(ik\frac{-2xX - 2yY + x^2 + y^2}{2z_p}\right)dx\,dy \end{aligned} \tag{6}$$

Question 2 **Pre-lab**	**When is the Fresnel the approximations valid?** Understanding the range of validity for a particular model is every bit as important as putting error bars on the data. Being able to analyze an approximation and make physical sense of when the approximation is and is not valid in your experiment is a sophisticated and important skill. a. What quantity has to be small for the approximation to be valid? Hint: You will have to consider the next higher set of terms (quartic terms) in the Taylor Series in Equation 5, and write a condition for when those quartic terms are small (i.e., negligible). b. At what distance z_p between the slit and the detector do you expect the Fresnel prediction to break down as a function of aperture width a and wavelength λ? Does the Fresnel approximation break down at larger or closer distances?

FRAUNHOFER APPROXIMATION

The Fraunhofer approximation involves the additional assumption that we can ignore the exponentiated quadratic terms with $\exp ikx^2/z_p$ and $\exp iky^2/z_p$ Equation 6. In the Fruanhofer approximation then approximate the exponent ikr_p as

$$kr_p \approx kz_p + \frac{k(X^2+Y^2)}{2z_p} + \frac{k(-2xX-2yY)}{2z_p} \tag{7}$$

And the full Fraunhofer diffraction integral becomes

$$\psi_{diff}^{\text{Fraunhofer}}(\vec{R}) = \frac{1}{i\lambda z_p}\exp(ikz_p)\exp\left(ik\frac{X^2+Y^2}{2z_p}\right)\int \psi_{trans}(x,y)\exp\left(ik\frac{-xX-yY}{z_p}\right)dx\,dy \tag{8}$$

Question 3 **Pre-lab**	**When is the Fraunhofer the approximations valid?** a. Again, considering the Taylor series expansion in Equation 5, what quantity has to be small for the approximation to be valid? b. At what distances z_p between the slit and the detector do you expect the Fraunhofer prediction to break down?

MODELING THE MEASUREMENT

Question 4 **Pre-lab**	**Justifying the use of a 1D model of diffraction** Using a photodetector and a 1D translation stage makes it fairly simple to measure a 1D diffraction pattern. This question explores the conditions in which a 1D model of diffraction can describe an actual experiment in the lab. a. Under appropriate conditions the 2D Fresnel and Fraunhofer diffraction integrals in Equations 6 and 8 separate into a product of 1D integrals. What does this assume about the aperture shape and the electric field ψ_{trans} after the aperture? b. If you want to observe the 1D pattern, what does this assume about the detector? c. Write the 1D version of the Fresnel and Fraunhofer integrals.
Question 5	**Quantitative test of diffraction in the Fraunhofer regime** One of the trickier parts of this lab is getting good quantitative agreement on the amplitude of the measured signal, not just the pattern. A convenient normalization for the electric field is that $$P = \int \lvert\psi(x,y)\rvert^2 dxdy$$ So that the electric field squared, integrated over the aperture gives a power. So we can normalize the initial field immediately after the aperture $\psi_{trans}(x,y)$ so that the total power transmitted through the aperture is given by $$P_{trans} = \int_{\text{aperture}} \lvert\psi_{trans}(x,y)\rvert^2 dxdy$$ Then the power on the detector in the diffraction plane is given by $$P_{detector} = \int_{\substack{\text{detector}\\ \text{aperture}}} \left\lvert\psi_{diff}(X,Y)\right\rvert^2 dXdY$$ a. **Your experimental goal is to predict and measure the Fraunhofer diffraction pattern** getting agreement in the fringe spacing and overall shape AND in the actual magnitude of the measured power. • This will require that you measure the profile of the beam incident upon the slit, appropriately normalize the field $\psi_{trans}(x,y)$, and predict the measured power level using the known aperture of your photodetector. • Your prediction may benefit from simplifying the diffraction pattern into the product of two 1D integrals.

COMPARISON OF THE FRESNEL AND FRAUNHOFER PREDICTIONS

Question 6	**Comparing Fresnel and Fraunhofer predictions for a uniformly illuminated slit.** a. Write a Mathematica function to evaluate the 1D Fresnel and Fraunhofer diffraction integrals for a uniformly illuminated slit. • The Mathematica function `UnitBox` may be useful for describing the initial electric field. • You measure intensity, not electric field, so your predicted diffraction patterns should be proportional to $\|\psi\|^2$ b. Create plots of the predicted diffraction patterns and show they agree in the regime where Fraunhofer is valid. c. Make a plot of the Fresnel and Fraunhofer predictions at the particular length scale where the Fraunhofer approximation should become inaccurate? (See question 2) d. How much smaller does z_p have to be so that the plots of the Fresnel and Fraunhofer patterns start to deviate significantly?

EXPERIMENT TO MEASURE THE TRANSITION BETWEEN FRAUNHOFER AND FRESNEL DIFFRACTION.

Question 7	**Design an experiment to measure the transition between the Fresnel and Fraunhofer regions.** You may find it helpful to make some predictions prior to taking measurements. These predictions can guide your experimental design as you consider the following factors: a. What slit width should you use for the aperture? Why? b. What spatial resolution do you need from your detector? c. What distances will you measure the pattern at? Why? d. What incident beam do you want on the aperature slit? How will you measure this? e. Should you worry about the overall normalization of the diffraction pattern, or just the position dependence? Carry out your experiment. You will probably find it useful to use a variable width slit and the NI USB-6009 DAQ to quickly record your data.
Question 8	**Compare your results with predictions of the model.** • Does your data agree with your prediction within your measurement uncertainty (i.e., does your prediction provide a “good fit”)? • Are there any significant sources of random error? • Are there any significant sources of systematic error? For example, this could include uncertainty in model parameters, or idealizations in the model which are not very applicable in the actual experiment. For any sources: • Quantify the uncertainty in the model parameter or in what way the idealization breaks down. • What kind of effect would this error source have on your predictions?

	If you want to get better agreement, how can you modify your setup or model? Do it.

WEEK 2: FOURIER OPTICS

GOALS FOR WEEK 2

After completing the second week of this lab you should be able to compute the Fourier transform of the electric field in order to predict a diffraction pattern. You observe and computationally model a number of interesting patterns for unusual apertures.

CONNECTION BETWEEN FRAUNHOFER DIFFRACTION AND THE FOURIER TRANSFORM

The Fraunhofer diffraction integral given in Eq. (8) bears a lot of similarity to the Fourier Transform given below in Eq. (9)

$$H[k'] = \int_{-\infty}^{\infty} h(x) e^{i2\pi k' x} dx \qquad (9)$$

Where we can interpret x as the position in the diffraction plane, and k' as the spatial frequency of the diffraction

Question 9 **Pre-lab**	Comparing the definition of the Fourier transform given in Eq. 9 with with the Fraunhofer diffraction integral a. What is $h(x)$? b. What is k'? c. What other normalization factors are there?

COMPUTING FOURIER DIFFRACTION PATTERNS EFFICIENTLY

The integral method of computing diffraction patterns works, but you may have noticed the integrals can take a long time to compute. We can take advantage of the connection between the Fourier Transform and Fraunhofer diffraction and use a very efficient method of calculating Fourier Transforms called the Fast Fourier Transform (FFT). The next part of the lab will have you calculate diffraction patterns using the Fast Fourier Transform in a variety of situations, including the prediction of 2D diffraction patterns.

There are two challenges to using the FFT to model your experiment and make predictions:

1. You must appropriately discretize the electric field, so it is only specified on a 1D grid for 1D diffraction patterns, or a 2D grid for 2D diffraction patterns. Figure 1 shows wave fronts (red) incident upon a slit (black). The electric field at the aperture is evaluated along a discrete grid of points.
2. You must interpret the Fourier transform in a way that connects your predictions to your data. This lab will give you a lot of “hands-on” experience with FFTs of a variety of functions.

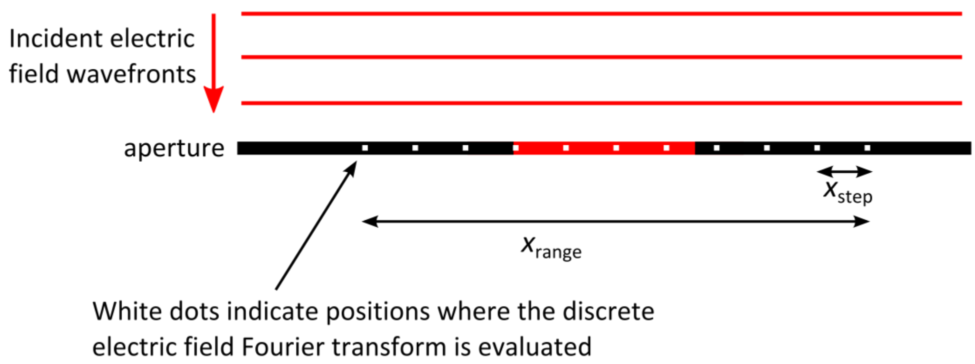


Figure 3: Discretizing the electric field for a one-dimensional diffraction pattern

FAST FOURIER TRANSFORM IN THE CASE OF A UNIFORMLY ILLUMINATED SLIT

Before you start the lab you should grab the notebook fft_helpers.nb from the course website. The notebook contains four helper functions:

1. `fft[list]`
   - Takes the Fast Fourier Transform of the 1D array called `list`. FFT normalization matches MATLAB and Python's numpy package.
2. `fftfreq[n, d]`
   - Generates a list of frequencies to pair with the `fft` function. `n` is the length of the list, and `d` is the sample rate of the original data.
3. `fftshift[list]`
   - Used to reorder a FFT or frequency list to go from the most negative to most positive frequency, with zero in the middle.
4. `fftshift2D[list]` Same as fftshift, but for 2D lists.
   - Same as fftshift, but for two dimensional Fourier transforms. It will be used for predicting and plotting the 2D diffraction patterns at the end of the lab.

The following set of code gives an example of how to compute the diffraction pattern of a uniformly illuminated slit. It will generate the correct pattern for a Fraunhofer diffraction pattern, but the x-axis will be in "spatial frequency" (like Hertz is to time).

```
a = 1*10.^-4;  (* a is full slit width*)
xstep = 10.00*^-6;  (*grid size for aperture*)
xrange = 1*10.^-3; (*Range in the aperture plane*)
slitField = Table[UnitBox[x/a],{x,-xrange,xrange,xstep}];
(*Uniform electric field across slit of width a*)
posdata = Range[-xrange,xrange,xstep]; (*Position data, only
needed for plotting*)


ListPlot[ Transpose[{posdata,slitField}],
```

```
      PlotRange->All,
      FrameLabel->{"Position in aperture plane, x (m)",
            "Electric field (a.u.)"}]

(*Electric field pattern in arbitrary field units*)
diffField = fft[slitField]
(*Intensity pattern in arbitrary intensity units*)
diffPattern = Abs[fft[slitField]]^2;
(*create the spatial frequency array*)
freq = fftfreq[Length[diffPattern],xstep];

(*combine the spatial frequency and intensity data and plot*)
ListPlot[Transpose[{freq,diffPattern}],
      PlotRange->All,
      FrameLabel->{"Spatial frequency (m^-1)",
            "Intensity (a.u.)"}]
```

Question 10	**Use the example code above to predict a 1D diffraction pattern from a uniformly illumiated slit.** a. How does the x-step size change the calculated pattern? Why do you think this is? b. How does the x-range used for calculating the fast Fourier transform affect he calculated pattern? Why?

Question 11	**Comparing the FFT result with the Fraunhofer diffraction integral.**
	a. Using your result from Question 9 where you compared k' in the Fourier Transform with the parameters in the diffraction integral, how would you rescale the x-axis into X so you can compare with the Fraunhofer diffraction?
	b. For the same slit width, make a plot comparing
	    a. The rescaled FFT result (you might need to adjust the vertical normalization)
	    b. The Fraunhofer integral
	    c. Actual data (you should have data from the Week 1 of the lab)
	c. Are there any differences between the FFT result and the diffraction integral? Does it depend on the x-range or x-step?
	d. Which calculation is faster? The FFT or the Fraunhofer Integral? You can get exact timing information using the `Timing` function in Mathematica. Check the help entry for `Timing` to get more information.

Question 12	**Using the FFT to predict a more complicated scenario:** You should demonstrate good quantitiative agreement between the prediction and measurments, meaning you will have to understand your laser beam characteristics and your aperture dimensions accurately. Choose from among the following scenarios, or create your own. • A non-uniformly illuminated slit, meaning the Gaussian beam is roughly same width as the slit or smaller, and possibly off-center on the slit. • A transmission diffraction grating (available as a 35 mm slide). Mathematica's `SquareWave[{y1,y2}, x]` generates a square wave that alternates between `y1` and `y2` with unit period.

GETTING TO THE FRAUNHOFER REGIME BY ADDING A LENS

The connection between the Fraunhofer diffraction pattern and the spatial Fourier transform of the electric field is very useful for basic and applied research. There is an entire sub-field of optics called "Fourier Optics" dedicated to exploiting this relationship to do interesting science. However, the Fraunhofer diffraction pattern only occurs when viewed in the far-field, which can be a considerable distance from the aperture depending on dimensions. It would be awkward to always design our optical systems so large that the patterns form in the far field. Luckily, there is another way…

Question 13	**A colleague knowledgeable in the study of light** says that if a lens is added after the aperture, then the Fraunhofer diffraction pattern will appear in a plane a focal length from the lens (called the focal plane). Answer the following questions through experimental testing, keeping good records of your observations, and being quantitative when possible. a. Is your colleague right? Why or why not? b. Does it matter how far the lens is from the aperture? • For example, whether the lens form an image of the aperture or not. • Does the lens need to be in the Fraunhofer regime? c. What factors affect the size of the diffraction pattern?

2D FOURIER DIFFRACTION PATTERNS

So far, all of your diffraction calculations have been for one dimensional diffraction patterns, but all of the patterns you observed are two dimensional. One of the primary reasons for this is computational efficiency. The Fraunhofer diffraction patterns calculated in week one were doable for a graph of intensity versus x, but would be very time consuming to compute a plot of intensity versus x and y. Using the fast Fourier Transform method can speed up the calculation drastically, so that 2D patterns are computationally reasonable.

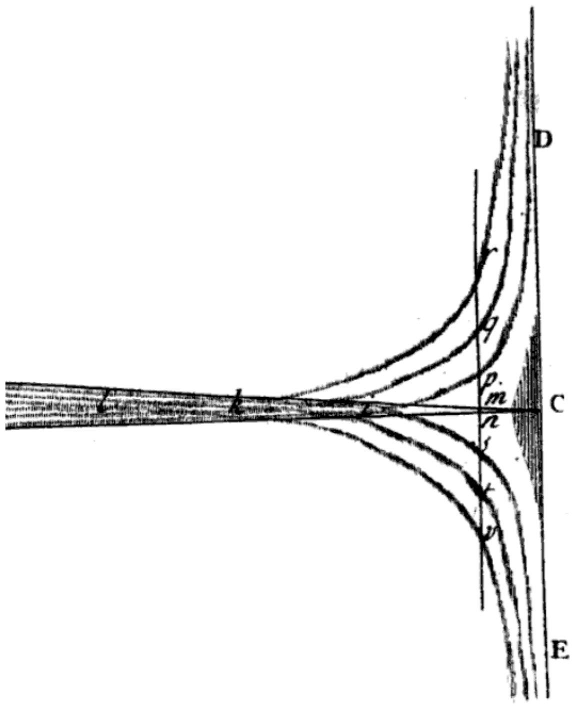
The procedure for generating a 2D pattern is as follows:

1) Generate a 2D array (a.k.a "list" in Mathematica) where each element is the value of the electric field over a grid. This can be done in two ways:
   - Generate the array of values in Mathematica (good for when you have a specific functional form for the electric field, like Gaussian)
   - Use a scale image of the aperture to represent the discretized electric field. Then import the image into Mathematica (good for uniformly illuminated apertures of complicated shapes).
2) Take the Fast Fourier Transform (FFT) of the array.
   - You will need the `fft` and `fftshift2D` functions defined in fft_helpers.nb
3) Approximate the visual appearance of the pattern in three steps:
   - Intensity is proportional to the absolute value of the FFT value squared…
   - Add a background signal which washes out the weaker fringes.
   - The eye responds to intensity on a roughly logarithmic scale, so we take the log of the dataset with offset.

4) Rescale the x and y axes in physically meaningful units.
    - `ArrayPlot` has a `DataRange->{{xmin,xmax},{ymin,ymax}}` option which changes the x- and y-axes from array index into the range of your own choosing.
5) Plot the adjusted intensity.

A Mathematica notebook with two examples of computing 2D FFTs is provided on the course website.

Question 14	**Two dimensional pattern from a Gaussian laser beam incident upon a slit.** For the parallel slits studied earlier in the lab, we only considered the 1D diffraction patterns. This was justified in special cases when it happened that the diffraction integral separated into a product of 1D integrals over x and over y. Now you will use the 2D FFT method to predict the full 2D pattern of a Gaussian beam upon a parallel slit. a. Predict the 2D diffraction pattern of a Gaussian laser beam incident upon a parallel slit. Make sure you convert the x and y axes to the appropriate units. b. Set up the measurement and compare with prediction. Demonstrate agreement by taking a picture of the diffraction pattern next to a ruler to give a sense of scale in the image. You will need to be quantitative about both the beam size and aperture to demonstrate good agreement.
Question 15	**Creating and modeling cool 2D diffraction patterns from apertures of your choice.** A variety of 35 mm projector slides are available with different patterns of circles and slits. Compare your experimentally observed pattern with a prediction made using the FFT method. Demonstrate quantitative predictions and an experimental comparison for at least one of the apertures. • When photographing a pattern, show a ruler for comparison to give the scale. Two perpendicular rulers could define to axes. • Quantitatively measuring the dimensions of the apertures can be done using the measuring microscope. • For complicated apertures, you may also find it helpful to use a program like Microsoft Paint or Paintbrush for Mac to draw a black and white image of the aperture, which can be imported into Mathematica. The black and white color information in the image is used to represent the discretized electric field.

Question 16	**Newton's Double Knife Edge Challenge** Although Thomas Young gets a lot of credit for doing the double slit experimenting and firmly establishing the wave nature of light, up-ending Newton's particle model of light, Newton also had compelling evidence. The following graph is a set of hyperbolic diffraction fringes that Newton sketched and published in his book *Opticks* (free copy available on Google books). The fringes were created by diffraction of sunlight from a narrow wedge formed from two knife blades. Newton did not have a detailed wave model of light to use for predictions, and he did not have the benefit of a monochromatic light source like your HeNe laser. So although the pattern is explained by a wave model, Newton did not have any model to predict the result, and saw it more as a curiosity rather than evidence for a wave model of light.2  a. Construct a narrow wedge and observe the diffraction pattern. How does it compare with Newton's sketch? b. Is the Fraunhofer diffraction pattern observed in the focal plane of a lens the same as the pattern at far distances? c. Predict the far-field diffraction pattern from a wedge using the 2D Fourier Transform method. Does it agree with the observed diffraction pattern?
Question 17	**Industrial espionage challenge** You have hired by an established greeting card company that wants to expand its product line to include giftwrapping and decorative bows. An industrial spy working for the current giftwrapping industry leader in giftwrapping was paid to steal top secret samples of an incredibly shiny and decorative new "paper" which is about to hit the market before a major gift-giving holiday. Your job is to use your expertise in diffraction of light to reverse engineer the new paper, and provide the greeting card company with the specifications necessary to duplicate the design.

PROJECT IDEAS

1) **Fourier Filtering.** The basic setup covered here can be slightly extended so that the image represents a filtered version of the original object. This requires a second aperture to adjust filter Fourier components by blocking them in the diffraction plane (which is the FFT of the initial aperture). The filtering aperture could be a fixed aperture, or it could have a number of controllable pixels. A controllable filter is often called a spatial light modulator.

REFERENCES

1. Hecht, Eugene *Optics* 4th edition (2002) Chapters 10 covers diffraction and Chapter 11 covers Fourier Optics. We have copies in the lab.
2. Goodman, Joseph W. *Introduction to Fourier Optics* 3rd edition